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Bi-Objective Flow Shop Scheduling with Equipotential Parallel Machines

Malhotra, K.[1,2], Gupta, D.[1], Goel, S. *[1], and Tripathi, A.K.[1]

[1]*Department of Mathematics, Maharishi Markandeshwar (Deemed to be University), Ambala, India*
[2]*Department of Mathematics, D.A.V. College (Lahore), Ambala City, India*

E-mail: sonia.mangla14@gmail.com
**Corresponding author*

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Abstract

Scheduling is very important concept in each and every field of life especially in case of manufacturing process. Actually, a production schedule is the time table that tells the time at which an assignment will be processed on various machines. The Schedule also gives the information about starting and finishing of a work on one machine. This paper also deals with the theory of Scheduling. The main attraction of this study is the optimization done on like Parallel machines with the help of Fuzzy Processing Times. Here the problem of optimization on Two Stage Flow Shop Model has been taken into consideration. This paper reveals an algorithm using Branch and bound method for scheduling on three like parallel machines available at initial stage and solo machine at next stage having processing period of all works as fuzzy triangular numbers involving transportation time from first stage to second stage. Algorithm provides an optimal sequence of jobs for minimizing make span as well as the unit operational cost of each job on all three parallel machines. Numerical example has also been discussed for elaborating this situation. The proposed model is the extension of model presented by Deepak Gupta and Sonia Goel [18].

Keywords: scheduling; parallel machines; fuzzy triangular number;branch and bound; make span.

1 Introduction

Modern world is a technological world. Technology is the base of success of each business and each business or industry is the backbone of economy of each country. So, for increasing GDP of a aountry, promotion of industries is the most essential part. As each business works on the concept of planning and scheduling. Planning includes understanding the demand of customer, market trend to any resource and availability of resources. Scheduling means proper allocation of all available and limited resources over time (Pinedo). Scheduling in business plays the most effective role. It reserves time as well as participate in growth of the productivity of system. Actually, in scheduling theory, researchers expresses some problem related to industrial growth or related to any real world problem into the form of a model and then include different conditions and apply methodologies of scheduling theory for scheduling tasks in proper way for satisfying some measures of effectiveness. Measures of effectiveness involves many criterias like cost minimization, time minimization, proper utilization of all jobs in a specific period of time, completing project up-to due date and so many. These models may be of many types, if these are dependent on machines then these are called One Stage, Two Stage, Three Stage and so on but if these models are dependent on the sequence of operations of jobs then these are of the types like Single Machine Model, Flow Shop Model, Job Shop Model , Open Shop Model and Parallel Machine Model. In these models, one thing is very important which is the utilization time of all jobs is. If this period of processing a job is fixed that means one job will be processed for how much time on one machine, then these kinds of problems will be classified as deterministic flow shop problems. In case processing times are not known exactly then the problems are considered under the preview of fuzzy logic. All these problems in scheduling theory are converted into many type of models which are discussed above but the Parallel machine scheduling model has been extensively explored optimization problem. Parallel Machine Scheduling Models are also of three types: Identical Parallel Machine Model, Uniform Parallel Model, and Unrelated Parallel Machine Model.

In this paper we are working on Identical Parallel Machine Model. The standard like parallel machine scheduling models involve the arrangement of n- autonomous tasks say on a set of m alike parallel machines and each task i is having fixed processing time but in today's real-life situations as uncertainty is all around, the precised value for processing times cannot be assigned. Here the concept of fuzziness arises. Zadeh (1965) first introduced the theory of fuzziness. This present research is working on the model of like parallel Flow Shop Model and considering two constraints Fuzzy utilization time and Transportation time. Here two performance measures which are minimization of total elapsed time and minimization of unit operational cost have been checked with the help of Branch and Bound method. Our research is motivated by the research of Deepak Gupta and Sonia Goel (Optimization of Production scheduling in two stage Flow Shop Scheduling Problem with m equipotential machines at first stage) by adding the concept of Fuzziness on processing times of jobs with parallel identical machines). Any industry like garment factory or manufacturing industry or textile industry depends on its customers. Customer demand in market plays a vital role in the success of a business. Each business works on the principle of sales and purchase. Purchasing of raw material and selling of ripe products promote a business and increase the profits in case of excess demand of a product. Due to limited resources or less availability of raw materials, all demands are not fulfilled. So, more resources like machines for manufacturing these products should be set up for same type of work so that more production should be possible. Here the need for parallel or equipotential machines arises. Suppose we consider the case of garment factory, here it is not possible to prepare whole cloth only on one machine because it has to pass through many processes like fabric selection, patternmaking, grading, marking, spreading, cutting, bundling, sewing, pressing and folding. Each process demands more than one machine so that more units are produced in minimum time and in minimum cost. So, the same type of machines is the essential need so that work or assignment for fulfilling customer's need should be completed in minimum time. Along with these situations, one more problem is existing in today's world that is nobody wants to wait for the things he wished for. He will move wherever he finds it quickly. So, this situation demands production of more items in negligible time. So, the time in each process of production is not fixed. Hence our model is fuzzy in nature and considering the concept of parallel machines for the same type of assignment. As each process takes some time before going to second machine, hence transportation time from initial level to next has been involved also. As unit cost of each process must also be optimized. Hence our model is bi-objective.

2 Related Works

Nature has blessed man with its resources as a boon. Now it depends on a man how he will use these resources so that no body remains untouched with the showring of these boons as they are consolidated. So, here scheduling helps in proper use of these limited resources. Actually, scheduling is the process of developing schedules and shows a plan for the timings of certain assignments. This is the most important decision in optimal utilization of limited available limited resources. A researcher must be focused not only on attaining the optimality of solution but he should be focused on practical and economical application of that solution also.

Several studies combine the scheduling and resource allocation problems that occur in flexible manufacturing environments. The first study in this direction was done in 1954 [21]. Johnson presented a simple decision rule leads to optimal scheduling of items (jobs) minimizing the total elapsed time for complete operation considering two stage and three stage flow shop problems. Then Smith [33] went for choosing 'best' schedule in his research. He worked on single stage production system by considering the concept of due date of a job and showed that maximum tardiness of job completion can be minimized by arranging the job according to their due date on which the task is scheduled to be completed. Little [25] introduced a methodology involving Branch-Bound strategy for those scheduling models which are considering set-up costs, associated with travelling salesman problem. Giglio and Wagner [13] worked on three stage mathematical model and applied different computational methods like integer programming, linear programming with answers rounded to integers, heuristic algorithms and random sampling. These all methods were applied on 100 sets of data randomly generated and finally concluded that a heuristic algorithm based on Johnson's method is beneficial for small subsets of considered data. Next, Dudek and Teuton [11] gave algorithm for the m stage flow shop problem and genearlized it with some theory using combinatorial analysis of algorithm and gave one example for this. After that, Karush [23] gave an counter example taking the case of 3 jobs, 3- machines problem in which he showed that the method proposed by Dudek [11] fails to produce an optimal sequence. Then, Smith and Dudek [34] modified the paper of Dudek and Teuton [11] by showing that minimization of total idle time on the last machine also minimizes total time required. After that, a decomposition approach for the machine scheduling problem was introduced by Ashour [3]. Here the original problem was broken into smaller sub-problems and then compared the efficiency of results by obtaining with decomposition approach and complete enumeration. Next, Ignall and Schrage [19] worked on two and three stage flow shop problem with the help of Branch and Bound method and compared with the methods used by Giglio and Wagner [13] and used by Dudek and Teuton [11] and found the better results. One mathematician used Roy's graphical theory and developed Branch Bound algorithm for 3 stage flow shop problems [26]. Then Mc Mohan et al. [28] introduced rules for ordering machines and listing jobs prior to application of branch bound algorithm and gave new method of obtaining lower bounds. Two types of bounds were introduced, one is job based bound and other is machine based bound.

In classical situations, most of problems were dealt with single machines. Azizoglu M, Kondacki, Suna and Omer [4] started working on Single machines and checked performance measures of weighted sum of earliness and tardiness penalities. Chen and Bulfin [7] considered single machine problem and examine the complexity of scheduling problem when more than one performance measures are included which are tardiness, number of tardy jobs, flow time and so on. Ahmed Abu Cenna and Mario T. Tabucanon [5] studied the bi-parameter with the help of identical parallel processors. These two criterias are minimizing total flow-time (measure of inventory) and minimization of maximum tardiness (measure of customer service). He compared five different dispatching rules and showed that best result is given by Wassenhove and Gelders. Due to rising demand of customers in market, similar machine scheduling has an important role in industrial field. Parallel machines scheduling problem is very useful in today's technological and rising world. This Parallel Scheduling means setting up of same type of machines on one stage so that same type of job can be processed parallelly on different equipotential machines for saving time of processing. All machines have the same function. Actually, a major benefit of multiple machine system is to take the whole system as one aggregate facility. In this direction, Cheng and Sin [8] worked on the concept of parallel machine scheduling. Here the major research results in deterministic parallel machine scheduling theory. Mokotoff [29] also dealt with many situations considering the cases of parallel machine scheduling problems. Various circumstances happen in a manufacturing plant, like tool loading, procedure assignment, tool swapping and arrangement. Crama [10] discussed these type of optimization problems arising in robotic trade systems, and offers mathematical models and solution practices. Chung [9] also worked on identical parallel machines.

Also, concepts related to uncertainty for completing a project exist in market. The word used to describe uncertainty regarding period of fulfilling a task in the theory of scheduling is 'fuzzy'. L.A. Zadeh [39] firstly uncovered the mystery of uncertainty by throwing concept of fuzzy theory. He presented it in the form of formula of mathematics checking uncertainty in daily life. He extracted out Fuzzy set theory which is very important concept of Artificial Intelligence. Fuzzy set theory is easy to understand and analogous to human reasoning. McCohon [27] cast-off average of fuzzy sets. He presented that a fuzzy number can represent the processing time interval exactly as fuzzy number is itself a generalized interval. In his work, he used triangular and trapezoidal, two types of fuzzy numbers. He modified Johnson's [21] and Ignall and Schrage's algorithms [19] for accepting job processing times. Ishibuchi, Murata and Lee [20] described the FSSP taking fuzzy numbers. They considered processing times of jobs as fuzzy numbers with the help of fuzzy arithmetic. The scheduling criterias are also calculated as fuzzy numbers and lastly applied multi objective genetic algorithm as a heuristic method for solving the problem. A researcher named A. B. Chandramouli [6] also gave a new simple heuristic algorithm for 3-machine, n-job flow shop scheduling in which jobs are involved with weights and breakdown intervals of machines are also given. Next, Singh T.P. and Gupta Deepak [31] gave probabilistic models of FSSP. After that, the concept of rental cost is introduced by Gupta and Sharma [14, 30]. In this paper, they developed a heuristic algorithm to minimize rental cost under special rental policy. Next, in the next research Deepak Gupta [15] took the concept of set up time under fuzzy environment and involved single transporting facility also. An algorithm was presented here for the minimization of flow time and verified by mathematical theorem.

Transportation time concept also plays an important role in scheduling theory. Transportation time is the time duration taken by each job for going on to Second machine after completing on first machine. So, this is the most important constraint which should be optimized. Many theories regarding this also have been given in the area of scheduling. Studies related to rental cost criteria involving transportation time were also done [32]. After that, concept of branch bound technique taking transportation time was also applied [16]. Next, in this paper, Kayvanfar [24] took the problem of single machine and applied heuristic and two metaheuristic approaches for solving the considered problem in which the performance measures which are to be optimized are taken as tardiness and earliness. A very important concept of learning effect was also introduced by Yeh et al. [38] in 2014 where two heuristic algorithms were proposed and computational experiments were also been conducted to check the performance of algorithms. Next, Fazel [12] discovered a bi-objective scheduling problem with controllable processing times on identical parallel machines.

This research is mainly encouraged by the approval of the just-in-time (JIT) philosophy on identical parallel machines in terms of bi-objective approach, where the job processing times are controllable. The aim of this study is to simultaneously minimize (1) total cost of tardiness, earliness as well as compression and expansion costs of job processing times and (2) maximum completion time or make-span. Next, Teymourian [36] developed an enhanced intelligent water drops algorithm for scheduling of an agile manufacturing system. He worked on two stage, one is machining and other is assembly and took parallel identical machines on assembly stage. His water drops algorithm is based on new swarm nature inspire optimization algorithm and artificial immune system algorithm has also been proposed to solve the considered problem. In next research, Amin Aalaei [1] added one more concept as compared to Fazel's research [12], that concept is work-in-process as this concept is needed in many industrial applications. Gupta and Goel [17] also worked on three stage FSSP considering equipotential machines Models. In next research, Lang Wu et al. [37] minimized the concept of long run expected make-span of stochastic customer order. Here explicit expressions are provided for deterministic workload case and many production requirement cases. In 2021, Jovanovic [22] worked on unrelated parallel machines with sequence dependent set up times. Here he used meta-heuristic approach to solve the proposed problem in which he included Ant Colony Optimization (ACO), Worm Optimization (WO), Greedy Randomized Adaptive Search Procedure (GRASP) and found Fixed Set Search is best suited here. Many authors worked on different optimization models like Sulaiman et al. [35] worked on Productivity Cost Model. Next, Fatma Adam and Nasruddin Hassan [2] worked on group decision methodology.

3 Materials and Methods

3.1 Problem Formulation

Suppose the given situation is of considering n jobs for processing on two types of machines naming R and S. Type R has three equipotential machines and Type S has single machine where $(\alpha_i, \beta_i, \gamma_i)$ and $(\theta_i, \eta_i, \xi_i)$ denote utilization time of i^{th} job on R and S machines in the order. The processing of each job is not compulsory on each of the equipotential parallel machines of type R, it may be possible that one job is processed only on first or second or third like machine of type R and then after completing the process of various phases of one task on first machine, the task will be transferred to second machine for further processing. The goal of the exploration of the proposed topic is to obtain the superlative scheduling of tasks for optimizing the elapsed time as well as unit cost of processing of all the jobs in an optimal manner to like/equipotential machines. The matrix form of model is presented as follows:

3.2 Mathematical Model

Explanation of Table 1:
Here in Table 1, there are n jobs which are to be processed on two types of machines namely R

Table 1: Matrix form of the model.

Jobs	Machine R	Processing time of job on machine R	Transportation Time	Processing time of job on machine S
I	R_1 R_2 R_3	Fuzzy time (r_i)	T_i	Fuzzy time (s_i)
(1)	β_{11} β_{12} β_{13}	$(\alpha_1, \beta_1, \gamma_1)$	T_1	$(\theta_1, \eta_1, \xi_1)$
(2)	β_{21} β_{22} β_{23}	$(\alpha_2, \beta_2, \gamma_2)$	T_2	$(\theta_2, \eta_2, \xi_2)$
(3)	β_{31} β_{32} β_{33}	$(\alpha_3, \beta_3, \gamma_3)$	T_3	$(\theta_3, \eta_3, \xi_3)$
.	.	.	.	
.	.	.	.	
.	.	.	.	
.	.	.	.	
(n)	β_{n1} β_{n2} β_{n3}	$(\alpha_n, \beta_n, \gamma_n)$	T_n	$(\theta_n, \eta_n, \xi_n)$
Available time	t_1 t_2 t_3			

and S. Type R has three identical parallel machines namely R_1 , R_2 , R_3 and Type S has one single machine. The unit operational cost of each job i on Machine R is given as β_{ij}. The processing time of each job i on machine R is given as fuzzy triangular numbersr_i denoted by $(\alpha_i, \beta_i, \gamma_i)$. Similarly the processing time of each job i on machine S is given as fuzzy triangular number si denoted by $(\theta_i, \eta_i, \xi_i)$. Transportation time from first machine to second machine is given as T_i. Available time of like parallel machines R_1 , R_2 , R_3 are given as t_{11} , t_{12} and t_{13} respectively.

3.3 Notations

i	:	job index
j	:	machine index
$(\alpha_i, \beta_i, \gamma_i)$	:	processing time taken as fuzzy triangular number of job i on machine R.
$(\theta_n, \eta_n, \xi_n)$	:	processing time of job i on machine S considered as fuzzy triangular number.
R_j	:	like machines for machine R, j = 1,2,3.
β_{ij}	:	unit operational cost of job i (1,2, 3,.....,n) on Rj like machine j =1,2,3.
t_{1j}	:	total available time of like machines.
T_i	:	transportation time from first machine to second machine.

3.4 Assumptions:

1. All the tasks are independently processing.
2. Jobs are allowed to process partially.
3. Setup time is ignored.
4. Transfer time from initial step to next step has been counted.
5. It is not necessary that all jobs will be processed on all like/parallel machines. One job may be completed by working only on first type or second or solely on third machine of type R

only.

6. Each job has different operating cost.
7. The starting time of working of all the equipotential parallel machines may be matched.
8. Equipotential machines are in process for indeterministic phase of time.
9. The phase of non-availability of machines is not allowed in this situation,that is machines are available throughout the period.
10. Pre-emption is not considered here.

4 Algorithm

We give the solution of the above problem in four steps. In the First Step, we find out the mean high ranking using Yager's formula of the fuzzy processing time of each job on various machines. In Second Step, we create virtual machines. In Third Step, we apply MODI method for optimization of operation time of all the tasks on parallel/like machines. In Fourth Step, we discover the optimal classification of tasks using Branch and Bound method.

Step 1: Apply Yager's formula for finding crisp value which is, crisp

$$(\breve{A}) = h(\breve{A}) = r_i' = (3\beta_i + \gamma_i - \alpha_i)/3 \tag{1}$$

where $(\alpha_i, \beta_i, \gamma_i\ )$ denotes fuzzy processing time of i^th job on machine R.

$$s_i' = (3\eta_i + \xi_i - \theta_i)/3 \tag{2}$$

where $(\theta_i, \eta_i, \xi_i)$ is fuzzy processing time of ith job on machine S.

Step 2: Create two virtual machines M and N having processing time of i^th job as

$$m_i = r_i' + T_i \tag{3}$$

and

$$n_i = s_i' + T_i \tag{4}$$

Step 3: Estimate the best suitable sharing of unit cost of each job to like/equipotential machines by using MODI method.
Check the condition

$$\sum_{j=1}^{3} t_{1j} = \sum_{i=1}^{n} m_i \tag{5}$$

if this is true then it is balanced problem and go to next point below.

Apply Modified Distribution Method to attain the greatest probable sharing of unit cost of each job to like machines. If is unbalanced situation that is either

$$\sum_{j=1}^{3} t_{1j} > \sum_{i=1}^{n} m_i or \sum_{j=1}^{3} t_{1j} < \sum_{i=1}^{n} m_i \tag{6}$$

then add dummy job n+1 with the processing time on all machines having values zero or we can add dummy like parallel machine R4 with unit operational cost having value zero.

Step 4:

(a) Apply Branch and Bound method and get optimal scheduling of jobs. Apply the formula

$$l' = \max(\sum_{i=1}^{n} R_{ij}) + \min_{i \in J'_t}(n_i) \tag{7}$$

$$l'' = \max_{i \in J'_t}(R_{ij}) + \sum_{i=1}^{n} n_i \tag{8}$$

where Jt denote jobs of Branching Tree and $J't$ denote jobs which are not considered under branching tree.

(b) Calculate $l = \max\{l', l"\}$

(c) Find out l for the n numbers, starting from 1 and going to n respectively, which are the node points of scheduling tree.

(d) Check the topmost minimum charge. Then take $(n-1)$ arrangements, compute l, initializing from the above chosen vertex as the first element of optimal sequence. Next, again continue the same process and find out l for $(n-2)$ sub nodes. Ongoing in the same way, ultimately, we will reach at the end of the branching tree with the required nodes which will represent the jobs of optimal scheduling sequence.

(e) Prepare in/out table for the best choice of that sequence which has been obtained in the above section for finding the least time to finish all the jobs.

4.1 Flow Chart Supporting Methodology:

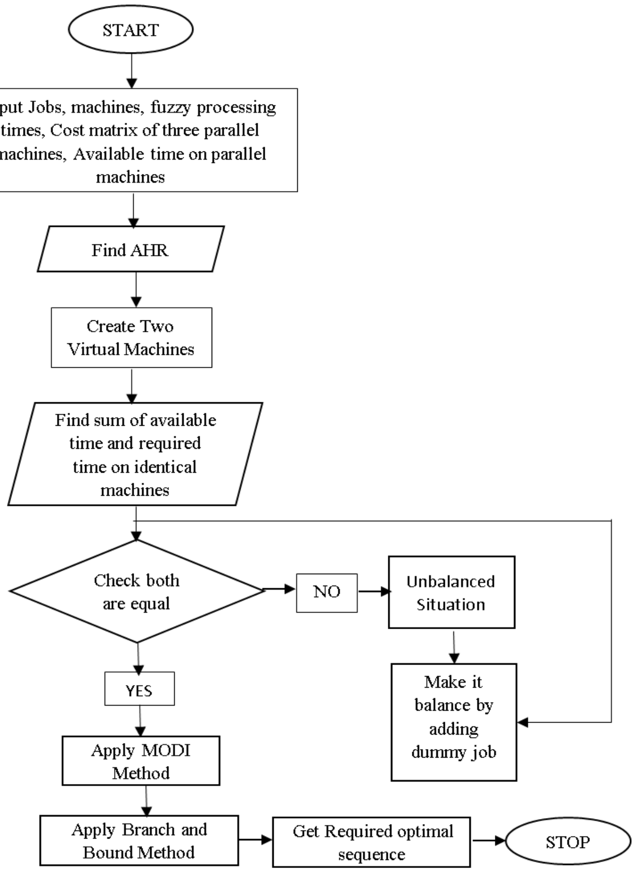

Figure 1: Flow chart of model.

4.2 Verification Using Illustration:

4.2.1 Illustration 1:

The following illustration is best suited for the above given methodology. Let us suppose that the situation is of processing four jobs on machine R and then on S where three like parallel machines of type R and single machine of type S exist. The processing of jobs will be in order on R and S respectively. Fuzzy triangular numbers have been taken into consideration as the processing time of jobs on these two types of machines. Operating cost of all like machines is also taken into under consideration. Total available time for each of the parallel machine is also given. The purpose of the above methodology is to explore the most suitable arrangement of tasks so as to optimize the elapsed make span as well as to decrease the entire expenses of doing all the jobs.

Table 2: Numerical problem.

Jobs	Machine R				Transportation Time	Machine S
I	R_1	R_2	R_3	Processing time(r_i) $(\alpha_i, \beta_i, \gamma_i)$	T_i	Processing time(s_i) $(\theta_i, \eta_i, \xi_i)$
1	5	6	4	(3,4,5)	8	(2,3,4)
2	1	2	3	(5,6,7)	11	(6,7,8)
3	2	3	1	(6,7,8)	12	(8,9,10)
4	3	4	7	(7,8,9)	10	(10,11,12)
Available Time t_{1j}	35/3	30/3	30/3			

Explanation of Table 2 Here four jobs (i=1,2,3,4) are taken into consideration for processing on two types of machines, one is R and second is S. R has three like parallel machines R1, R2, R3 and S has one single machine. Unit operational costs of job 1 are 5, 6 and 4 under R_1, R_2, R_3 respectively. Unit operational costs of job 2 are 1, 2, 3 under R_1, R_2, R_3 respectively. Similarly, Unit operational costs of job 3 are 2, 3, 1 and of job 4 are 3, 4 and 7 under R_1, R_2, R_3 respectively. Processing Time of job 1 is (3,4,5) on machine R and (8,9,10) on machine S. Similarly processing time of job 2 on machine R is (5,6,7) and on machine S is (11,12,13).Processing time of job 3 on machine R is (6,7,8) and on machine S is (7,8,9).Processing time of job 4 on machine R is (7,8,9) and on machine S is (15,16,17).Transportation Time from machine R to Machine S of job 1, 2, 3, 4 are given as 8, 11, 12 and 10 respectively. Now the objective is to find optimal sequence of jobs in order to minimize total elapsed time which is called make-span also and to minimize unit operational cost also.

Solution: We solved the problem in four steps:

Step 1: According to Step One, use defuzzi-fication formula, then fuzzy triangular numbers will be changed into crisp values and the reduced problem will be as follows:

Table 3: Numerical problem after applying Yager's formula.

Jobs	R1	R2	R3	r_i'	Ti	s_i'
1	5	6	4	14/3	1/3	11/3
2	1	2	3	20/3	2/3	23/3
3	2	3	1	23/3	5/3	29/3
4	3	4	7	26/3	4/3	35/3
Available Time t_{1j}	35/3	30/3	30/3			

Explanation of Table 3 Here all fuzzy triangular processing times have been changed to crisp processing times.
As (3, 4, 5) is changed to (3*4 +5-3)/3 =14/3
(5, 6, 7) is changed to (3*6 +7 -5)/3 = 20/3
(6, 7, 8) is changed to (3*7+8-6)/3 =23/3
(7, 8, 9) is changed to (3*8+9-7)/3 = 26/3
Similarly, (2, 3, 4) is changed to 11/3
(6, 7, 8) is changed to 23/3

(8 , 9, 10) is changed to 29/3
(10, 11, 12) is changed to 35/3

Table 4: Virtual machines M and N.

Jobs/ Machine	Machine M			Processing time of machine M	Processing time of machine N
I	R_1	R_2	R_3	m_i	n_i
1	5	6	4	15/3	12/3
2	1	2	3	22/3	25/3
3	2	3	1	28/3	34/3
4	3	4	7	30/3	39/3
t_{1j}	35/3	30/3	30/3	95/3	

Explanation of Table 4 We have created virtual machines M and N where the processing time of i^th job is m_i on Machine m and n_i on Machine N. Here $m_i = r'_i + T_i$ and $n_i = T_i + s'_i$

Table 5: Optimal allocation of processing time on equipotential machines.

Jobs	R_1	R_2	R_3	N
1	0	15/3	0	12/3
2	22/3	0	0	25
3	13/3	15/3	0	34/3
4	0	0	30/3	39/3

Explanation of Table 5 According to Step 3, MODI method has been applied to generate Table 5.

Next, According to Step 4, Apply branch and bound method and calculate lower bounds for the first branch of tree and get the extreme values

Lower bound (1) = maximum (35/3 + 25/3, 15/3 +110/3) = maximum (60/3, 125/3) = 125/3

Similarly,

Lower bound (2) = maximum (35/3 +12/3, 22/3 +110/3) = maximum (47/3, 132/3) =132/3

Lower Bound (3) = maximum (47/3, 125/3) = 125/3

Lower Bound (4) = 140/3.

Here the lowest bound is 125/3 that is connected with job 1 or 3. Hence, without loss of generality, we put job no. 1 at the primary place in the ideal required sequence and next continue to explore another job of the ideal sequence.

Now, determine nodes for the second branch of tree.

Lower Bound (12) = maximum (69/3,132/3) = 132/3

Lower Bound (13) = maximum (60/3,140/3) = 140/3

Lower Bound (14) = maximum (60/3,140/3) = 140/3

Here lowest bound is 132/3 and the required subsequence at this stage is (12). Hence, we put task 2 at second place in the ideal sequence. Putting the task 1 and task 2 at initial and next positions correspondingly, now next move to the third task to be put in the ideal sequence. Compute the nodes for third branch of tree

Lower Bound (123) = maximum (74/3,145/3) = 145/3

Lower Bound (124) = 140/3

Here lowest bound is 140/3 and subsequence associated with this bound is (124). Hence 1, 2,

4 tasks are getting first, second and third positions for required sequence correspondingly and the last job 3 will automatically come to be on the last position. Finally, the required scheduled order is (1243).

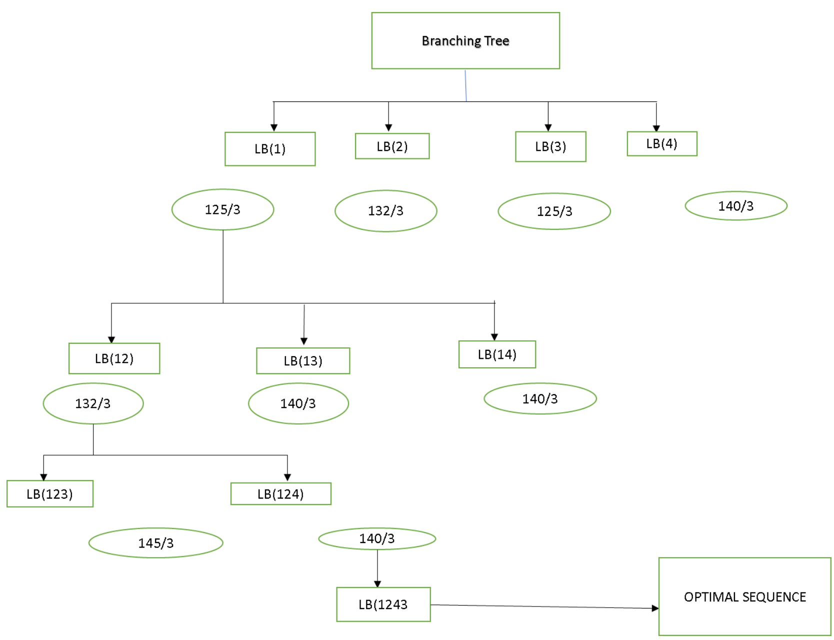

Figure 2: Branching tree.

Now after finding the optimal sequence of jobs, we will generate in-out table for checking out for minimal make-span.

Table 6: In - out table for best possible sequence.

Jobs	R_1	R_2	R_3	N
1	-	0-15/3	-	15/3-27/3
2	0-22/3	-	-	27/3-52/3
4	-	-	0-30/3	52/3-91/3
3	22/3-35/3	15/3-30/3	-	91/3-125/3

Explanation of Table 6 Hence according to Table 6, required elapsed time = 125/3 and minimum unit operational cost is given by the formula

$$6 * \frac{15}{3} + 1 * \frac{22}{3} + 2 * \frac{13}{3} + 3 * \frac{15}{3} + 1 * 0 + 7 * \frac{30}{3} = 131 Rs. \tag{9}$$

where 6 is unit operational cost of job 1 on machine R2 and 15/3 is the assigned optimal time to job 1 on machine R_2 , 1 is unit operational cost of job 2 on machine R_1 and 22/3 is the assigned optimal time to job 2 on machine R_1 ,7 is unit operational cost of job 4 on machine R_3 and 30/3 is assigned optimal time to job 4 on machine R_3 ,2 is unit operational cost of job 3 on machine R_1 and 13/3 is assigned optimal time to job 3 on machine R_1 and 3 is unit operational cost of job 3 on machine R_2 , 15/3 is assigned optimal time to job 3 on machine R_2.

4.2.2 Illustration 2.

The following illustration is best suited for the above given methodology. Let us suppose that the situation is of processing four jobs on machine R and then on S where three like parallel machines of type R and single machine of type S exist. The processing of jobs will be in order on R and S respectively. Fuzzy triangular numbers have been taken into consideration as the processing time of jobs on these two types of machines. Operating cost of all like machines is also taken into under consideration. Total available time for each of the parallel machine is also given. The purpose of the above methodology is to explore the most suitable arrangement of tasks so as to optimize the elapsed make span as well as to decrease the entire expenses of doing all the jobs.

Table 7: Numerical problem.

Jobs	Machine R				Transportation n Time	Machine S
I	R_1	R_2	R_3	Processing time(r_i) ($\alpha_i, \beta_i, \gamma_i$)	T_i	Processing time(s_i) (θ_i, η_i, ξ_i)
1	1	2	1	(3,5,7)	1	(2,3,5)
2	3	3	2	(1,3,6)	2	(2,5,7)
3	4	5	1	(1,2,5)	3	(4,6,8)
4	3	3	2	(4,6,9)	1	(1,3,5)
5	1	1	2	(1,3,5)	2	(2,4,8)
Available Time t_{1j}	35/3	40/3	31/3			

Explanation of Table 7 Here five jobs (i=1,2,3,4,5) are taken into consideration for processing on two types of machines, one is R and second is S. R has three like parallel machines R_1, R_2, R_3 and S has one single machine. Unit operational costs of job 1 are 1, 2 and 1 under R_1, R_2, R_3 respectively. Unit operational costs of job 2 are 3, 3, 2 under R_1, R_2, R_3 respectively. Similarly, Unit operational costs of job 3 are 4, 5, 1 ,unit operational cost of job 4 are 3, 3 and 2 under R_1, R_2, R_3 respectively and unit operational cost of job 5 are 1,1, 2 under R_1, R_2, R_3 respectively.. Processing Time of job 1 is (3,5,7) on machine R and (2,3,5) on machine S. Similarly processing time of job 2 on machine R is (1,3,6) and on machine S is (2,5,7).Processing time of job 3 on machine R is (1,2,5) and on machine S is (4,6,8).Processing time of job 4 on machine R is (4,6,9) and on machine S is (1,3,5). Processing time of job 5 on machine R is (1,3,5) and on machine S is (2,4,8).Transportation Time from machine R to Machine S of job 1 ,2,3,4,5 are given as 1 ,2,3,1and 2 respectively. Now the objective is to find optimal sequence of jobs in order to minimize total elapsed time which is called make-span also and to minimize unit operational cost also.

Solution: We solved the problem in four steps:

Step 1: According to Step One, use defuzzi-fication formula, then fuzzy triangular numbers will be changed into crisp values and the reduced problem will be as follows:

Table 8: Numerical problem after applying Yager's formula.

Jobs	R1	R2	R3	r'_i	T_i	s'_i
1	1	2	1	19/3	1	12/3
2	3	1	2	14/3	2	20/3
3	4	5	1	10/3	3	22/3
4	3	3	2	23/3	1	13/3
5	1	1	2	13/3	2	18/3
Available Time t_{1j}	35/3	40/3	31/3			

Explanation of Table 8 Here all fuzzy triangular processing times have been changed to crisp processing times.
As (3, 5, 7) is changed to (3*5 +7-3)/3 =19/3
(1, 3, 6) is changed to (3*3 +6 -1)/3 = 14/3
(1,2,5) is changed to (3*2+5-1)/3 =10/3
(4, 6, 9) is changed to (3*6+9-4)/3 = 23/3
(1,3,5) is changed to (3*3+5-1)/3=13/3
Similarly, (2, 3, 5) is changed to 12/3
(2, 5, 7) is changed to 20/3
(4, 6, 8) is changed to 22/3
(1, 3, 5) is changed to 13/3
(2,4,8) is changed to 18/3

Table 9: Virtual machines M and N.

Jobs/ Machine	Machine M			Processing time of machine M	Processing time of machine N
I	R_1	R_2	R_3	m_i	n_i
1	1	2	1	22/3	15/3
2	3	1	2	20/3	26/3
3	4	5	1	19/3	31/3
4	3	3	2	26/3	16/3
5	1	1	2	19/3	24/3
t_{1j}	35/3	40/3	31/3	106/3	

Explanation of Table 9 We have created virtual machines M and N where the processing time of i^{th} job is m_i on Machine m and n_i on Machine N. Here

$$m_i = r_i' + T_i \tag{10}$$

and

$$n_i = T_i + s_i' \tag{11}$$

Table 10: Optimal allocation of processing time on equipotential machines.

Jobs	R_1	R_2	R_3	N
1	0	15/3	0	12/3
2	22/3	0	0	25
3	13/3	15/3	0	34/3
4	0	0	30/3	39/3

Explanation of Table 10 According to Step 3, MODI method has been applied to generate Table 10.

Next, According to Step 4, Apply branch and bound method and calculate lower bounds for the first branch of tree and get the extreme values
Lower bound (1) = maximum (26/3 + 16/3, 22/3 +112/3) = maximum (42/3, 134/3) = 134/3

Similarly,

Lower bound (2) = maximum (26/3 +15/3, 20/3 +112/3) = maximum (41/3, 132/3) =132/3
Lower Bound (3) = maximum (41/3, 131/3) = 131/3
Lower Bound (4) = 125/3.
Lower Bound (5) = 131/3.

Here the lowest bound is 125/3 that is connected with job 4. Hence, we put job no. 4 at the primary place in the ideal required sequence and next continue to explore another job of the ideal sequence.
Now, determine nodes for the second branch of tree.
Lower Bound (41) = maximum (50/3,147/3) = 147/3
Lower Bound (42) = maximum (41/3,133/3) = 133/3
Lower Bound (43) = maximum (41/3,143/3) = 143/3

Lower Bound (45)= maximum (41/3,132/3)=132/3

Here lowest bound is 132/3 and the required subsequence at this stage is (45). Hence, we put task 5 at second place in the ideal sequence. Putting the task 4 and task 5 at initial and next positions correspondingly, now next move to the third task to be put in the ideal sequence. Compute the nodes for third branch of tree

Lower Bound (451) = maximum (52/3,147/3) = 147/3
Lower Bound (452) = 152/3
Lower Bound (453) = 143/3

Here lowest bound is 143/3 and subsequence associated with this bound is (453). Hence 4, 5,3 tasks are getting first, second and third positions for required sequence correspondingly and choose next job of sequence. Now, compute the nodes for fourth branch of tree.

Lower (4531) = maximum (52/3, 147/3) = 147/3
Lower (4532) = maximum (41/3 , 152/3) = 152/3
Here lowest bound is 147/3 associated with 4531 subsequence. So, subsequence chosen from here is 4-5-3-1.Finally, optimal
sequence is 4-5-3-1-2.
Now after finding the optimal sequence of jobs, we will generate In-Out table for checking out for minimal make-span .

Table 11: In - out table for best possible sequence.

Jobs	R_1	R_2	R_3	N
4	0-13/3	13/3-14/3	14/3-26/3	26/3-42/3
5	-	14/3-33/3	-	42/3-66/3
3	-	-	26/3-45/3	66/3-97/3
1	13/3-35/3	-	-	97/3-112/3
2	-	33/3-53/3	-	112/3-138/3

Explanation of Table 11 Hence according to Table 11, required elapsed time = 138/3 hrs.
And minimum unit operational cost is given by the formula

$$1 * \frac{22}{3} + 1 * \frac{20}{3} + 1 * \frac{19}{3} + 3 * \frac{13}{3} + 3 * \frac{1}{3} + 2 * \frac{12}{3} + 1 * \frac{19}{3} = \frac{146}{3} Rs; \tag{12}$$

where 1 is unit operational cost of job 1 on machine R_1 and 22/3 is the assigned optimal time to job 1 on machine R_1 , 1 is unit operational cost of job 2 on machine R_2 and 20/3 is the assigned optimal time to job 2 on machine R_2 ,1 is unit operational cost of job 3 on machine R_3 and 19/3 is assigned optimal time to job 3 on machine R_3 ,3 is unit operational cost of job 4 on machine R_1 and 13/3 is assigned optimal time to job 4 on machine R_1 and 3 is unit operational cost of job 4 on machine R_2 , 1/3 is assigned optimal time to job 4 on machine R_2 , 2 is unit operational cost of job 4 on R_3 and 12/3 is assigned optimal time of job 4 on R_3 and 1 is unit operational cost of job 5 on R_2 and 19/3 is assigned optimal time of job 5 on R_2.

5 Data Collection

The data of above two illustrations has been randomly generated for the completion of above purpose and for verification of proposed algorithm.

6 Conclusions

According to literature study, here above we explained different type of scheduling models like Single Machine Scheduling Model, Flow Shop Scheduling Model, Job Shop Scheduling Model, Open Shop Scheduling Model and Parallel Machine Scheduling Model, different restrictions like weightage of jobs, job block criteria, rental policies, and fuzzy nature of utilization time of jobs and so on. Different methodologies like heuristics approach, meta-heuristic approach in which swarm optimization, ant colony optimization method exist, applied on different models. But, In this our above exploration, we considered the model of two stage FSSP with three like parallel machines at initial level under uncertain environment with the aim of lessening the total elapsed time together with the minimization of cost of accessing all the jobs on equipotential machines. Here we used Branch and Bound method as compared to other methodologies its efficiency is improved by powerful lower and upper bounding procedures and some elimination mechanisms. So, branch and bound method gives exact solution. However, heuristics and meta-heuristics approach provide near optimal solution only. So, the methodology described here is the proposed and best suited method for extracting out minimum elapsed time as well as minimum operating cost of jobs. The work proposed here can be further extended by taking m parallel machines at initial level. The study can be extended by taking different type of fuzzy numbers as processing times. The work can be taken at advanced level by considering parallel machines at both the stages and even at third stage also.

7 Scope of the Research

The major uses of identical parallel machines in these models is to associate a perfect relationship between machines and jobs (tasks) and consequently arrange an optimal sequence of jobs on each machine for achieving some goals. The practical scope of our research can be seen in any patrol station where many patrol pumps are connected to the same storage tank. More pumps are used and the cost of delivering patrol will be minimized.

Conflicts of Interest The authors declare that there is no conflict of interest regarding the manuscript of this.

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